Quantum back-action in spinor condensate magnetometry

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We provide a theoretical treatment of the quantum backaction of Larmor frequency measurements on a spinor Bose-Einstein condensate by an off-resonant light field. Two main results are presented; the first is a "quantum jump" operator description that reflects the abrupt change in the spin state of the atoms when a single photon is counted at a photodiode. The second is the derivation of a conditional stochastic master equation relating the evolution of the condensate density matrix to the measurement record. We comment on applications of this formalism to metrology and many-body studies.

Atomic vapor magnetometers of spin-polarized alkali atoms are among the most sensitive field sensors demonstrated to date [1]. These magnetometers, based on the optical detection of Larmor precession, have demonstrated field sensitivities in the attoTesla/Hz$^{1/2}$ regime [2]. The use of optically trapped ultracold atoms as the sensing medium holds promise for magnetic microscopy at high spatial resolution as well as for significant improvements in field sensitivity via entanglement-assisted techniques [3–5]. Spinor Bose-Einstein condensates are particularly suited to field sensing applications due to their low spin relaxation rates and absence of density-dependent collision shifts [6].

The detection of Larmor precession and the subsequent estimate of the magnetic field relies on the dispersive interaction between the collective atomic spin and the optical field, followed by a quantum-limited measurement of the light. This interaction entangles the optical and atomic degrees of freedom. Measuring the light breaks this entanglement and must therefore cause a backaction on the atomic spins. The ultimate sensitivity of such atomic magnetometers is governed by the interplay between the projection noise of the atomic spins, photon shot noise and the quantum backaction due to the measurement of the optical field. Here, we provide a theoretical treatment of this backaction by means of a conditional stochastic master equation that relates the condensate evolution to the optical measurement record. One of the main results that distinguishes this work from similar studies, e.g. [7], is the derivation of a quantum jump operator reflecting the backaction on the atoms induced by the detection of a single photon.

Model: We consider a magnetometer consisting of a spinor Bose-Einstein condensate of $n \gg 1$ spin-1 bosons trapped in their motional ground state. Our formalism is well suited to both spinor gases with ferromagnetic (e.g. ^{87}Rb) and those with polar (e.g. ^{23}Na) ground states. We assume in this paper that the spatial extent of the gas is small enough that the single-mode approximation

(SMA) is valid. We also note that our results can be readily adapted to systems with different internal state spaces or expanded to incorporate spatial variations in the atomic spin field.

The atoms are prepared initially with high fidelity in the spin state,

$$|\psi\rangle = c_+|+1\rangle + c_0|0\rangle + c_-|-1\rangle \qquad (1)$$

in e.g. the z basis, with the normalization $\sum_i |c_i|^2 = 1$.

Before proceeding further, we introduce a "vector" notation over the spin indices for clarity and compactness, with the definitions

$$\mathbf{v} = (v_+, v_0, v_-), \qquad (2)$$
$$|\mathbf{v}| = v_+ + v_0 + v_-, \qquad (3)$$
$$\mathbf{v}! = v_+!v_0!v_-!, \qquad (4)$$
$$\mathbf{v}^{\mathbf{w}} = v_+^{w_+} v_0^{w_0} v_-^{w_-}, \qquad (5)$$

and the dot product defined in the usual way. For later use, we also define probabilities

$$p_i = |c_i|^2, \qquad (6)$$

where the state normalization implies that $|\mathbf{p}| = 1$. With this notation, the full n-particle state can be compactly expressed as

$$|\Psi_n\rangle = \hat{a}_\psi^{\dagger n}|0\rangle/\sqrt{n!} = \sum_{|\mathbf{j}|=n} \sqrt{\frac{n!}{\mathbf{j}!}} \mathbf{c}^{\mathbf{j}}|\mathbf{j}\rangle, \qquad (7)$$

where $\hat{a}_\psi^\dagger = \mathbf{c} \cdot \hat{\mathbf{a}}^\dagger$ is the Schrödinger field creation operator for state $|\Psi\rangle$, $|0\rangle$ is the vacuum, and $|\mathbf{j}\rangle$ is the state with j_i atoms in spin eigenstate i.

The spin state of the condensate is optically detected either by measuring the phase imprinted onto a short, coherent pulse of the light field by the condensate as in phase-contrast imaging [8, 9], or by monitoring the polarization of the light field as in polarization spectroscopy

(see, for example [10]). Without loss of generality, we assume the former method via balanced homodyne detection.

By introducing a general vector of coupling strengths $\mathbf{g}$, we can describe the light-atom interaction via the Hamiltonian

$$H_I = \hbar \mathbf{g} \cdot \hat{\mathbf{N}} \hat{b}^\dagger \hat{b}. \tag{8}$$

where $N_i = \hat{a}_i^\dagger \hat{a}_i$ and $\hat{b}$ is the light field annihilation operator. To derive this Hamiltonian, we have assumed that the probe field is detuned sufficiently far from the atomic resonance so that the excited atomic states can be adiabatically eliminated[11]. The effect of this Hamiltonian is to imprint a phase rotation on the optical field that contains information about the spin state of the atoms. The optical transit time is taken to be short enough that other contributions to the condensate's evolution may be neglected for now. The quantum state of the combined atomic spin and two output optical fields after passing through the final beam splitter, just before measurement, is therefore

$$\sum_{|\mathbf{j}|=n} \sqrt{\frac{n!}{\mathbf{j}!}} \mathbf{c}^{\mathbf{j}} |\mathbf{j}\rangle \otimes |\frac{A_0}{2}(1+ie^{-i\mathbf{G}\cdot\mathbf{j}})\rangle \otimes |\frac{A_0}{2}(1-ie^{-i\mathbf{G}\cdot\mathbf{j}})\rangle, \tag{9}$$

where A_0 is the dimensionless amplitude of the coherent optical pulse (i.e., the mean photon number is $|A_0|^2$), τ is the light-matter interaction time and $\mathbf{G} = \tau \mathbf{g}$. The probe beam's contribution to the output fields has been phase rotated through the unitary evolution given by the Hamlitonian (8). Importantly, unless the atoms are in the unlikely configuration of a Fock state of the spin field, this interaction entangles the optical field with the collective spin state of the atoms [12]. In the rest of this letter, we quantify the backaction induced on the condensate due to measurement of the light.

Assuming the photodectors have unit efficiency, the joint probability of obtaining photon counts C_+ and C_- at the first and second photodiodes, respectively, is

$$P(C_+, C_-) = e^{-|A_0|^2} \frac{(|A_0|^2/2)^{C_+ + C_-}}{C_+! C_-!} \sum_{|\mathbf{j}|=n} \frac{n!}{\mathbf{j}!} \mathbf{p}^{\mathbf{j}}$$
$$\times (1 + \sin \mathbf{G} \cdot \mathbf{j})^{C_+} (1 - \sin \mathbf{G} \cdot \mathbf{j})^{C_-}. \tag{10}$$

In the case $n|\mathbf{G}| \ll 1$ we find for the mean observed photon count difference the simple expression

$$\langle \Delta C \rangle \approx |A_0|^2 n \mathbf{G} \cdot \mathbf{p}$$
$$= |A_0|^2 n \left(G_0 + \frac{G_+ - G_-}{2} \langle F_z \rangle \right.$$
$$\left. + \frac{G_+ - 2G_0 + G_-}{2} \langle F_z^2 \rangle \right) \tag{11}$$

where $\Delta C = C_+ - C_-$ is the difference between photon clicks at the first and second photodiodes, and $\langle F_z \rangle = p_+ - p_-$ and $\langle F_z^2 \rangle = p_+ + p_-$ are the single atom spin expectation values.

The signal in Eq. (11) can be interpreted as follows. If the atoms are precessing in a magnetic field perpendicular to the light, the phase contrast signal has a large DC component proportional to the number of atoms, an AC signal at the Larmor precession frequency, and a third, smaller signal at twice the Larmor frequency. The amplitude of the respective contributions can be controlled by changing the polarization and/or the detuning of the probe light, thereby changing the relative values of the quantities $\mathbf{G}$ [8].

In an experiment, the number of condensed atoms is classically uncertain, so the spin state $|\Psi_n\rangle$ is replaced by a density matrix $\hat{\rho}$. Atom number in an experimental condensate obeys roughly Poissonian statistics:

$$\hat{\rho} = e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} |\Psi_n\rangle\langle\Psi_n| \tag{12}$$

where $\bar{n}$ is the mean atom number over many runs and $|\Psi_n\rangle$ is given by Eq. (7). Equation (11) is modified for this case by making the substitution $n \to \bar{n}$.

So far, we have considered the measurement of the condensate state by a pulse of light. To reach a continuous measurement limit, we instead consider coherent light with a photon flux rate per unit time f rather than an amplitude A_0, and examine the evolution of the system in a time interval δt such that $f\delta t \ll 1$. In this case, the probability of counting no photons at either detector is nearly unity, and any case where multiple photons are counted is negligible. By making the substitution $A_0 \to \sqrt{f\delta t}$ we can therefore model this continuous observation as a series of short, weak pulses. The free evolution of the atoms will be negligible on this timescale, i.e., between individual photon clicks.

In this limit, the detection of a photon by either detector (and no photon by the other) has the effect of acting on the density matrix with one of two jump operators:

$$\hat{\rho}(1,0) = \hat{J}_+ \hat{\rho}_{\mathrm{I}} \hat{J}_+^\dagger,$$
$$\hat{\rho}(0,1) = \hat{J}_- \hat{\rho}_{\mathrm{I}} \hat{J}_-^\dagger,$$
$$\hat{J}_\pm = \frac{\sqrt{f\delta t}}{2} \left(1 \pm ie^{-i\mathbf{G}\cdot\hat{\mathbf{N}}} \right). \tag{13}$$

The traces of these post-click density matrices represent the probabilities of the respective photodetections. Both probabilities are much less than one; under our assumptions, no click (and hence, no change except a small rescaling of $\hat{\rho}$) is by far the most likely outcome. In the standard theory of photodetection applied to a cavity with damping κ and single mode operator $\hat{a}$, the jump operator for a photon click is $\sqrt{\kappa\delta t}\hat{a}$[13]. In the present case, there are two major differences: First, no atoms are removed from the system, and hence the jump commutes with $\hat{\mathbf{N}}$. Second, as long as $\langle \mathbf{G}\cdot\hat{\mathbf{N}} \rangle \ll 1$, a single jump will have only a very small effect on the atomic state. This is to be expected – a single non-resonant photon passing through our apparatus ought not perturb the atoms much. In the next section, we combine this jump operator with the free spin evolution of the atoms to arrive at

a master equation describing the effects of the detection process.

Conditional stochastic master equation: In accordance with the experimental situation, we consider both the atom-light interaction and the Zeeman interaction induced by the ambient magnetic field,

$$H_Z = \hbar g_L \mu_B \mathbf{B} \cdot \hat{\mathbf{F}}, \tag{14}$$

where g_L is the Landé g-factor, μ_B is the Bohr magneton, $\mathbf{B}$ is the applied magnetic field – eventually the field to be detected – and $\mathbf{F}$ is the Schrödinger field spin operator. This Hamiltonian could be modified or replaced in more general settings, e.g. by inclusion of the quadratic Zeeman effect. We consider the dynamics of the system over a timescale $\Delta t \gg \delta t$ and an incident flux f such that a natural separation of scales occurs,

$$f \Delta t \sim \epsilon^{-1}, \tag{15}$$
$$g_L \mu_B \langle \mathbf{B} \cdot \hat{\mathbf{F}} \rangle \Delta t \sim \epsilon, \tag{16}$$
$$\langle \mathbf{G} \cdot \hat{\mathbf{N}} \rangle \sim \epsilon^2. \tag{17}$$

Crudely speaking, these scales posit a large photon number, small free evolution, and an even smaller effect from a single quantum jump, respectively.

Next, we consider the set of all possible photocounts $\{(C_+, C_-)\}$ and the probabilities of their detection over the interval Δt. The density matrix will be acted on by jumps $\hat{J}_\pm$ alternated with unitary evolution due to Eq. (14). Because we do not have access to the microscopic details of photon arrival order nor photon arrival times, we must sum (integrate) over the final density matrices for all possible arrival orders (times) consistent with a particular photocount for the interval Δt. This calculation is aided by the fact that the jump operators will commute with the unitary evolution operators at least to order ϵ^2. The resulting conditional density matrices are normalized by the probability of detection of their particular photocounts. See [11] for further details or [14] for a thorough treatment of a similar situation. We use the resulting probability distribution on $\{(C_+, C_-)\}$, and find that, to next to leading order in ϵ,

$$\langle C_\pm \rangle = V_\pm = \frac{f\Delta t}{2}(1 \pm \langle \mathbf{G} \cdot \hat{\mathbf{N}} \rangle), \tag{18}$$

that is, the mean equals the variance in the counts. The covariance between C_+ and C_- vanishes to this order. Also, we note that the means reproduce the signal seen in Eq. (11). Because the higher moments of the distribution agree with the equivalent Gaussian moments to next to leading order in ϵ, we are able to characterize the photocounts as independent Gaussian random variables,

$$C_\pm = \frac{f}{2}(1 \pm \langle \mathbf{G} \cdot \hat{\mathbf{N}} \rangle)\Delta t + \sqrt{\frac{f}{2}(1 \pm \langle \mathbf{G} \cdot \hat{\mathbf{N}} \rangle)}\Delta W_\pm, \tag{19}$$

where $\Delta W_\pm$ are independent Wiener increments, with

$$\langle \Delta W_\pm^2 \rangle_E = \Delta t,$$
$$\langle \Delta W_+ \Delta W_- \rangle_E = 0. \tag{20}$$

Here the subscript E refers to the ensemble average (that is, over many experimental runs) rather than the usual expectation value. These Wiener increments are Gaussian random variables of zero mean that are functions of time. In the continuum limit, $\Delta \to d$, these equalities are exact without ensemble averaging.

The result Eq. (19) is crucial to the derivation of the complete stochastic master equation. We apply the operators $\hat{J}_\pm$ each $C_\pm$ times to ρ, act on it by the free unitary evolution, divide by its trace, and take the continuum limit to obtain [11],

$$\frac{d\hat{\rho}}{dt} = \frac{i}{\hbar}[\hat{\rho}, H'] + \frac{\sqrt{f}}{2}(\mathbf{G} \cdot \hat{\mathbf{N}}\hat{\rho} + \hat{\rho}\mathbf{G} \cdot \hat{\mathbf{N}} - 2\langle \mathbf{G} \cdot \hat{\mathbf{N}} \rangle\hat{\rho})\xi_-(t), \tag{21}$$

where the modified Hamiltonian H' is

$$H' = \hbar\left(g_L\mu_B\mathbf{B} \cdot \mathbf{F} + \frac{1}{2}\mathbf{G} \cdot \mathbf{N}\left(f + \sqrt{f}\xi_+(t)\right)\right), \tag{22}$$

and $\xi_\pm(t) = \frac{dV_\pm}{dt}$ are uncorrelated Gaussian white noise functions related to the photocurrents, with $dV_\pm = (dW_+ \pm dW_-)/\sqrt{2}$.

Equations (21) and (22) succinctly encapsulate the full effects of the measurement scheme on the atoms. First, the Hamiltonian is modified by a steady, effective interaction resulting from Eq. (8), with the photon number operator replaced by a classical coherent flux. Shot noise in the photon flux leads to an additional random, but unitary, evolution. Finally, the second term in (21) is the non-unitary evolution induced by the measurement of $\mathbf{G} \cdot \hat{\mathbf{N}}$.

Examples: We first examine a very simple application of this master equation to illustrate its salient features in an analytically closed form before moving on to the more experimentally relevant case of the measurement of a magnetic field. Namely, we will assume that there is no external field, $\mathbf{B} = 0$, and coupling coefficients $G_+ = -G_- = G, G_0 = 0$. This means that the setup will measure only $\hat{F}_z$, i.e.,

$$\mathbf{G} \cdot \hat{\mathbf{N}} = G\hat{F}_z. \tag{23}$$

and the total Hamiltonian is quantum non-demolition (QND) for the observable $\hat{F}_z$.

We can derive equations of motion for the observables of the system by multiplying Eq. (21) by their operators and taking the trace. Because of the large number of atoms involved, we assume that the Gaussian state ansatz will be valid; thus, only equations for the observables, their variances, and their covariances are needed. The chain rule of Itô calculus (resulting from Eq. (20)) must be used when considering the derivatives of the (co)variances, as they are non-linear functions of the other variables. We obtain, for example, $d\langle \hat{F}_z \rangle/dt = G\sqrt{f}\xi_-(t)v_z$, where v_z is the variance of $\langle \hat{F}_z \rangle$, and $\dot{v}_z = -fG^2v_z^2$, which quickly integrates to
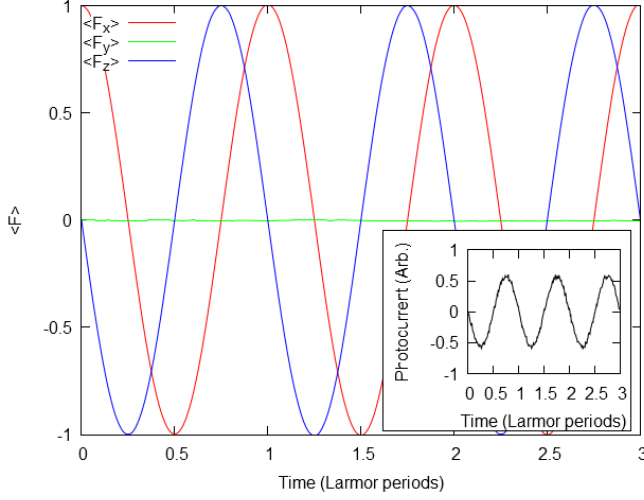
$$v_z(t) = (v_z(0)^{-1} + fG^2t)^{-1}. \tag{24}$$

FIG. 1: The evolution of the condensate's mean spin along each axis for $I = 0.01I_{\text{sat}}$, normalized by atom number, for a single simulated experimental run. The optical detection of Larmor precession has virtually no effect on the free dynamics. Insert: The normalized photocurrent difference oscillates at the correct frequency.

Note that determination of the mean value depends on the experimental record through $\xi_-(t)$, whereas the variance always decreases as the measurement continues. This should be unsurprising – continuous observation of $\hat{F}_z$ leads to greater certainty in its value. Over long times, the uncertainty can drop below the standard quantum limit. Next, we introduce a magnetic field to be measured, and present the results of a numerical integration
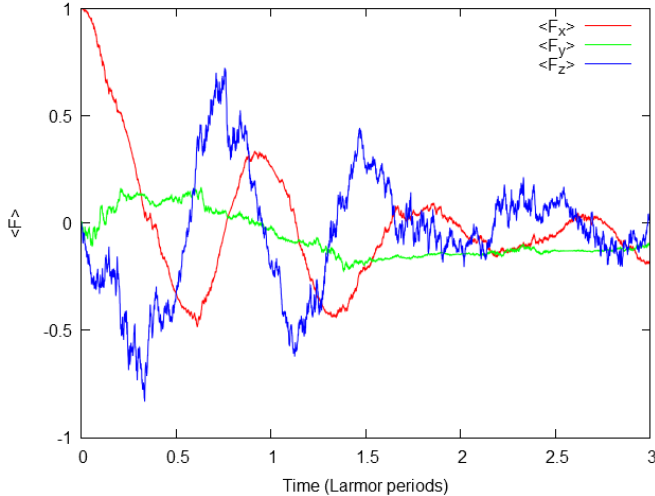

FIG. 2: The evolution of the condensate's mean spin projected along each axis for $I = I_{\text{sat}}$ for a single run. This stronger measurement causes rapid decay of the condensate's spin.

of the master equation over a single simulated experimental run. We measure this magnetic field by observing the spin in a transverse direction; the atoms' Larmor precession frequency reveals the field strength. We assume a condensate of 10^4 atoms confined within a transverse spatial extent of 15 μm in an ambient magnetic field of 1 mG in the y-direction. We apply an additional magnetic field $\frac{Gf}{2g_L\mu_B}$ in the negative z-direction; since G and f are well-controlled experimental parameters this should be possible to a high degree of accuracy. This additional field cancels out the classical portion of the light-matter interaction, though the photon shot noise fluctuations in Eq. (22) are still present. The probe is detuned by $2\pi\times$ 150 MHz below the $F = 1 \rightarrow F' = 2$ (D1) transition of ^{87}Rb, and interacts with the condensate for a total measurement time of 100 ms. The evolution of the condensate spin is shown for probe intensities of (Fig. 1) $I = 0.01I_{\text{sat}}$ and (Fig. 2) $I = I_{\text{sat}}$, where I_{sat} is the saturation intensity. The measurement strength can best be characterized by the dimensionless ratio $\frac{Gf}{\nu_L}$, where ν_L is the Larmor frequency. For the parameters listed, this ratio is 0.1 and 10., respectively. In the former case, the free evolution of the atoms is not noticeably perturbed; yet the photocurrent signal unambiguously oscillates at the Larmor frequency. This will allow an accurate determination of the applied field. On the other hand, as the probe intensity is increased and the measurement strength exceeds unity, the backaction-induced stochastic evolution overwhelms the free Larmor precession, resulting in a rapid decay of the transverse magnetization of the condensate in only a few oscillations. This crossover suggests a dynamical phase transition as one moves between the weak and strong measurement regimes.

Summary: We have provided a theoretical treatment of the quantum backaction due to the dispersive interaction between a spinor Bose-Einstein condensate and an off-resonant light field. In addition to being the basis for optical magnetometry using a Bose condensate, this interaction has also been shown to be a versatile quantum interface for quantum information processing and state engineering [15]. Straightforward additions to our model include a description of spatially inhomogeneous spin textures in the condensate, stroboscopic optical measurements of the condensate and the detection of time-varying magnetic fields. In addition to understanding the potential sensitivity of condensate-based magnetic field sensing, our formalism can also be applied to quantum-limited nondestructive imaging of magnetic textures in spinor condensates and the creation of novel many-body states via quantum non-demolition (QND) measurement [16].

tute of Technology for useful input on additional experimental considerations.

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